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Laura Menini, Corrado Possieri and Antonio Tornambè

***Abstract*—The main objective of this paper is to provide computational methods allowing one to characterize the observability of a class of systems having discontinuous right-hand side. In order to pursue this objective, it is first shown how tools borrowed from algebraic geometry can be used to characterize the observability of polynomial systems. Thus, by considering that elementary systems can be recast into polynomial form and that several systems having discontinuous right-hand side can be approximated by a 1-parameter family of elementary systems, such tools are adapted to deal with a class of elementary systems having discontinuous right-hand side. The key advantage of the proposed method is that it allows one to use classical tools, such as high-gain observers, to design state observers for systems having discontinuous right-hand side.**

I. Introduction

In several applications, unmeasurable state variables of a plant have to be estimated from the available measurements. Examples of practical applications are the detection and isolation of faults [1], [2], the estimation of unknown physical parameters from limited measurements [3], and the estimation of the attitude of a rigid body from inertial measurements [4] (see also [5]).

Due to this wide interest in observation problems, a large research effort has been spent to characterize the observability of dynamical systems. If the plant has linear, time-invariant dynamics, its observability can be easily characterized by using linear algebra tools [6], [7], [8], whereas the problem of characterizing the observability of a plant with nonlinear dynamics is much more challenging [9]. Some necessary conditions and a sufficient condition for the observability of nonlinear systems are given in [10], whereas necessary and sufficient conditions for the observability of analytic and polynomial systems are given in [11]. Such conditions entails with the injectivity of the observability map of the system, that is the map relating the current state of the system with the output and its time derivatives.

Several methods have been proposed in the literature to design observers for nonlinear systems. Some remarkable examples are: the extended Kalman filter [12], high-gain observers [13], [14], [15], [16], discrete-time iterative schemes [17], and sliding mode observers [18].

The main objective of this paper is to provide computational tools that allow one to study the observability of a class of systems having discontinuous right-hand side [19]. In order to pursue this objective, it is first shown how some tools borrowed from algebraic geometry can be efficiently employed to determine whether there exists a rational left inverse of the observability map of a polynomial system. Secondly, it is shown that several nonlinear systems that arise in practical engineering applications can be recast into polynomial form, thus allowing one to directly use the tools developed for polynomial systems. Finally, by considering that several systems having discontinuous right-hand side can be approximated by a 1-parameter family of elementary systems, the tools developed for polynomial plants are adapted to characterize the observability of this family of systems, independently of the parameter governing the approximation.

The key advantage of the proposed method to characterize the observability of systems having discontinuous right-hand side is that it provides also an inverse of the observability map, thus allowing one to readily construct a state observer by using classical tools, such as high-gain observers.

Examples of applications of the proposed methods are given all throughout the paper in order to illustrate and corroborate the theoretical results.

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II. Analysis of the observability of polynomial systems through algebraic tools

The main objective of this section is to review some results given in [20], [21] in order to illustrate how some tools borrowed from algebraic geometry can be used to characterize the observability of (non-switching) non-linear, autonomous, polynomial systems. Such results are used in the subsequent Section III to characterize the observability of systems having discontinuous right-hand side.

Consider the following *non-linear polynomial system*:

$$\dot{x} = f(x), \qquad\qquad y = h(x), \tag{1}$$

where $x = [\, x_1 \ \ \cdots \ \ x_n \,]^\top$ is the state vector, $y \in \mathbb{R}$ is the output, $f \in \mathbb{R}^n[x]$ and $h \in \mathbb{R}[x]$. The *solution* of system (1) at time $t \in \mathbb{R}_{\geqslant 0}$ with initial condition $x_0 \in \mathbb{R}^n$ is denoted $x(t) := \phi_f(t, x_0)$ and is assumed to exist for all $t \in \mathbb{R}_{\geqslant 0}$ (in particular, it is assumed that solutions to system (1) do not blow up in finite time) and to be unique. Let $y^{(k)}(t) := \frac{\mathrm{d}^k y(t)}{\mathrm{d}t^k}$ denote the *k-th time-derivative of the output* y, $k \in \mathbb{Z}_{\geqslant 0}$ and let $y_{e,N}(t) := [\, y^{(0)}(t) \ \ \cdots \ \ y^{(N)}(t) \,]^\top$ for all $N \in \mathbb{Z}_{\geqslant 0}$. System (1) is *observable* if there do not exist two different initial states $x_1, x_2 \in \mathbb{R}^n$ with $x_1 \neq x_2$ such that $h(\phi_f(t, x_1)) = h(\phi_f(t, x_2))$ for all times $t \in \mathbb{R}_{\geqslant 0}$ [22]. By [11], since both f and h have entries being polynomials in x, system (1) is observable if the state $x(t)$ at time $t \in \mathbb{R}$

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can be expressed as a function of $y^{(0)}(t), \ldots, y^{(N)}(t)$, for some sufficiently large (but finite) $N \in \mathbb{Z}_{\geqslant n-1}$. In particular, defining the *successive directional derivatives of h along f* as $L_f^0 h(x) := h(x)$ and $L_f^{j+1} h(x) := (\frac{\partial}{\partial x} L_f^j h(x)) f(x)$, $j \in \mathbb{Z}_{\geqslant 0}$, and letting N be a sufficiently large integer, $N \geqslant n-1$, consider the *observability map of order N of system* (1),

$$O_N(x) := \begin{bmatrix} h(x) \\ \vdots \\ L_f^N h(x) \end{bmatrix}.$$

Note that, for linear systems of the form

$$\dot{x} = Ax, \quad y = Cx,$$

the map $O_N(x)$ is given by $[\ C^\top \ \ \cdots \ \ (CA^N)^\top\ ]^\top x$ and it is injective if and only if $N \geqslant n-1$ and the pair (A, C) is observable [7], [23].

The map $O_N(x)$ relates the current output and its successive time-derivatives with the state of system (1), *i.e.*,

$$y_{e,N}(t) = O_N(x(t)), \quad \forall t \in \mathbb{R}_{\geqslant 0}. \tag{2}$$

By Theorem 1 of [11], system (1) is observable if and only if there is $\bar{N} \in \mathbb{Z}_{\geqslant n-1}$ such that, letting $\mathcal{Y}_{\bar{N}} := O_{\bar{N}}(\mathbb{R}^n)$ be the image of $\mathbb{R}^n$ through $O_{\bar{N}}$, there is $K_{\bar{N}} : \mathcal{Y}_{\bar{N}} \to \mathbb{R}^n$ such that

$$y_{e,\bar{N}} = O_{\bar{N}}(K_{\bar{N}}(y_{e,\bar{N}})), \quad \forall y_{e,\bar{N}} \in \mathcal{Y}_{\bar{N}}. \tag{3}$$

Algebraic geometry tools can be used to characterize the observability of system (1). In particular, letting $N \in \mathbb{Z}_{\geqslant n-1}$ be fixed, consider the ideal $\mathcal{I}_N$ of $\mathcal{R}_a := \mathbb{R}[x_1, \ldots, x_n, y^{(0)}, \ldots, y^{(N)}]$ (briefly, $\mathcal{R}_a := \mathbb{R}[x, y_{e,N}]$),

$$\mathcal{I}_N := \langle y^{(0)} - L_f^0 h(x), \ldots, y^{(N)} - L_f^N h(x) \rangle, \tag{4}$$

that is the ideal generated by the constraints given in (2), which hold for all $t \in \mathbb{R}_{\geqslant 0}$. In view of Theorem 2 of [21], the set of all the polynomial constraints (usually referred to as *embeddings*) that exist among the time-derivatives of the output up to order N is given by $\mathcal{E}_N := \mathcal{I}_N \cap \mathbb{R}[y^{(0)}, \ldots, y^{(N)}]$, which is an ideal of $\mathcal{R}_b := \mathbb{R}[y^{(0)}, \ldots, y^{(N)}]$ (briefly, $\mathcal{R}_b := \mathbb{R}[y_{e,N}]$). In particular, consider the following proposition.

Proposition 1. *$\mathbf{V}(\mathcal{E}_N)$ is the Zariski closure of $\mathcal{Y}_N$,* i.e.*, $\overline{\mathcal{Y}}_N = \mathbf{V}(\mathcal{E}_N)$. Furthermore, $\mathbf{V}(\mathcal{E}_N)$ is* irreducible*,* i.e.*, if $\mathbf{V}(\mathcal{E}_N)$ is written in the form $\mathcal{V}_1 \cup \mathcal{V}_2$, where $\mathcal{V}_1$ and $\mathcal{V}_2$ are varieties, then either $\mathcal{V}_1 = \mathbf{V}(\mathcal{E}_N)$ or $\mathcal{V}_2 = \mathbf{V}(\mathcal{E}_N)$.*

By the Elimination Theorem [24], the reduced Gröbner basis $\mathcal{G}_{\mathcal{E}_N}$ of the ideal $\mathcal{E}_N$ with respect to the Lex order, with $y^{(0)} \succ \cdots \succ y^{(N)}$, is obtained by computing the reduced Gröbner basis $\mathcal{G}_{\mathcal{I}_N}$ of the ideal $\mathcal{I}_N$ with respect to the Lex order, with $x_1 \succ \cdots \succ x_n \succ y^{(0)} \succ \cdots \succ y^{(N)}$, and by retaining the entries that are independent of $x_1, \cdots, x_n$, *i.e.*,

$$\mathcal{G}_{\mathcal{E}_N} = \mathcal{G}_{\mathcal{I}_N} \cap \mathbb{R}[y^{(0)}, \ldots, y^{(N)}].$$

Hence, define the quotient ring $\mathcal{R}_c := \mathbb{R}[y^{(0)}, \ldots, y^{(N)}]/\mathcal{E}_N$ (briefly, $\mathcal{R}_c := \mathbb{R}[y_{e,N}]/\mathcal{E}_N$), let $\mathbb{K}$ be the field of the rational functions with numerator and denominator being polynomials in the ring $\mathcal{R}_c$, and define the ring $\mathcal{R}_d := \mathbb{K}[x_1, \ldots, x_n]$ (briefly, $\mathcal{R}_d := \mathbb{K}[x]$). Note that using $\mathcal{R}_d$ as ambient ring essentially means that the coefficients of all the polynomials in x have to be considered as rational functions whose numerator p_1 and denominator p_2 coerced into $\mathcal{R}_b$ are such that $p_i \% \mathcal{E}_N = p_i$, $i = 1, 2$. Thus, coerce the ideal $\mathcal{I}_N$ into $\mathcal{R}_d$ and consider the elimination ideals

$$\mathcal{K}_{N,j} := \mathcal{I}_N \cap \mathbb{K}[x_j], \quad j = 1, \ldots, n. \tag{5}$$

By Corollary 4 (p. 41) of [24], the ideal $\mathcal{K}_{N,j}$ is *principal*, *i.e.*, there exists $g_{N,j} \in \mathbb{K}[x_j]$ such that $\mathcal{K}_{N,j} = \langle g_{N,j} \rangle$. In particular, such a polynomial $g_{N,j}$ is unique up to multiplication by a non-zero element of $\mathbb{K}$. The following theorem shows how the polynomials $g_{N,j}$ can be used to characterize the observability of system (1).

Theorem 1. *There exists a rational map $K_N \in \mathbb{R}^n(y^{(0)}, \ldots, y_N)$ such that $y_{e,N} = O_N(K_N(y_{e,N}))$ for almost all $y_{e,N} \in \mathcal{Y}_N$ if and only if, letting $\{g_{N,j}\}$ be the reduced Gröbner basis of $\mathcal{K}_{N,j}$, one has that*

$$\mathrm{LT}(g_{N,j}) = x_j, \quad j = 1, \ldots, n. \tag{6}$$

Theorem 1 provides a computational tool to test the observability of system (1). In fact, if the condition given in (6) holds, then system (1) is "generically" observable, whereas, if such a condition does not, then there does not exist a rational inverse of the observability map $O_N(x)$. It is worth mentioning that all the computations required to determine the polynomials $g_{N,1}, \ldots, g_{N,n}$ can be easily carried out by using the CAS software `Macaulay2` [25].

Example 1. Consider the dynamics of the Lorenz oscillator:

$$\dot{x}_1 = 10(x_2 - x_1), \qquad \dot{x}_2 = x_1(20 - x_3) - x_2, \tag{7a}$$
$$\dot{x}_3 = x_1 x_2 - \tfrac{8}{3} x_3, \qquad y = x_1. \tag{7b}$$

Hence, let $N = 3$, compute

$$\begin{aligned} h(x) &= x_1, \\ L_f h(x) &= 10x_2 - 10x_1, \\ L_f^2 h(x) &= -10x_3 x_1 + 300x_1 - 110x_2, \\ L_f^3 h(x) &= -10x_2 x_1^2 + \tfrac{710}{3} x_3 x_1 - 5200x_1 + 3110x_2 - 100x_2 x_3, \end{aligned}$$

and define the ideal $\mathcal{I}_3$ of $\mathbb{R}[x_1, x_2, x_3, y^{(0)}, y^{(1)}, y^{(2)}, y^{(3)}]$,

$$\begin{aligned} \mathcal{I}_3 := \langle & y^{(0)} - L_f^0 h(x), y^{(1)} - L_f^1 h(x), \\ & y^{(2)} - L_f^2 h(x), y^{(3)} - L_f^3 h(x) \rangle. \end{aligned}$$

By computing the reduced Gröbner basis of the ideal $\mathcal{I}_3$ according to the Lex order, with $x_1 \succ x_2 \succ x_3 \succ y^{(0)} \succ y^{(1)} \succ y^{(2)} \succ y^{(3)}$, and retaining only the entries that are independent of x_1, x_2, x_3, one obtains the reduced Gröbner basis $\mathcal{G}_{\mathcal{E}_3} = \{\eta_3\}$ of the ideal $\mathcal{E}_3 := \mathcal{I}_3 \cap \mathbb{R}[y^{(0)}, y^{(1)}, y^{(2)}, y^{(3)}]$,

$$\begin{aligned} \eta_3 = {} & 30(y^{(0)})^4 + 3(y^{(0)})^3 y^{(1)} - 1520(y^{(0)})^2 + 88y^{(0)} y^{(1)} \\ & - 33(y^{(1)})^2 + 41y^{(0)} y^{(2)} - 3y^{(1)} y^{(2)} + 3y^{(0)} y^{(3)}. \end{aligned}$$

This shows that $y^{(0)}$, $y^{(1)}$, $y^{(2)}$, and $y^{(3)}$ are algebraically dependent functions of time. In particular, one has that

$$\eta_3(y^{(0)}(t), y^{(1)}(t), y^{(2)}(t), y^{(3)}(t)) = 0, \quad \forall t \in \mathbb{R}.$$

Thus, let $\mathcal{R}_c := \mathbb{R}[y^{(0)}, y^{(1)}, y^{(2)}, y^{(3)}]/\langle \eta_3 \rangle$ and let $\mathbb{K}$ be the field of the rational functions whose numerator and denominator are in $\mathcal{R}_c$. By coercing $\mathcal{I}_3$ into $\mathbb{K}[x_1, x_2, x_3]$ and computing the reduced Gröbner basis $\mathcal{G}_{\mathcal{K}_{3,j}}$ of $\mathcal{K}_{3,j} = \mathcal{I}_3 \cap \mathbb{K}[x_j]$, one obtains that $\mathcal{G}_{\mathcal{K}_{3,j}} = \{g_{3,j}\}$, $j = 1, 2, 3$, with

$$g_{3,1} = x_1 - y^{(0)}, \qquad g_{3,2} = x_2 - \tfrac{10y^{(0)}+y^{(1)}}{10},$$
$$g_{3,3} = x_3 - \tfrac{190y^{(0)}-11y^{(1)}-y^{(2)}}{10y^{(0)}},$$

whence, by Theorem 1, system (7) is almost globally observable. In particular, if $y^{(0)} \neq 0$, or, equivalently, if $x_1 \neq 0$, then the current state of system (7) can be expressed as

$$x = K_3(y_{e,3}) = \begin{bmatrix} y^{(0)} \\ \frac{10y^{(0)}+y^{(1)}}{10} \\ \frac{190y^{(0)}-11y^{(1)}-y^{(2)}}{10y^{(0)}} \end{bmatrix}.$$

Note that this expression for the inverse of the observability map O_3 is not unique. As a matter of fact, since η_3 vanishes identically along the trajectories of system (7), for each $H \in \mathbb{R}^3[y_{e,3}]$, one has that the map

$$\check{K}_3(y_{e,3}) = K_3(y_{e,3}) + \eta_3(y_{e,3})H(y_{e,3})$$

is such that $y_{e,3} = O_3(\check{K}_3(y_{e,3}))$, for all $y_{e,3} \in \mathcal{Y}_3 \setminus \mathbf{V}(y^{(0)})$.

Note that the map K_N obtained by using the tools given in Theorem 1 can be easily used to design an observer for system (1). In fact, letting $\hat{y}^{(0)}, \ldots, \hat{y}^{(N)}$ be estimates of the time derivatives of the output $y^{(0)}, \ldots, y^{(N)}$ (which can be obtained by using classical tools available in the literature such as high-gain observers [13], sliding mode differentiators [18], and super-twisting algorithms [26]), an estimate of the current state of system (1) is given by

$$\hat{x}(t) = K_N(\hat{y}_{e,N}(t)). \tag{8}$$

III. Immersion of discontinuous and elementary systems into polynomial form

The main objective of this section consists in showing that several systems that arise in practical engineering applications can be immersed into the polynomial form (1), thus making the tools given in Section II readily usable to characterize their observability.

Let $\mathbb{E}[\xi]$ be the set of all the *elementary functions* in ξ, *i.e.*, the set of all the functions $\alpha : \mathbb{R}^m \to \mathbb{R}$ that are solution of a scalar polynomial differential equation in α with coefficients in $\mathbb{R}[\xi]$. Some examples of elementary functions are

$$\begin{aligned} \alpha = \sin(\omega\,\xi),\ \alpha = \cos(\omega\,\xi) &\text{ satisfying } \alpha'' + \omega^2\alpha = 0, \\ \alpha = \exp(\lambda\,\xi) &\text{ satisfying } \alpha' - \lambda\alpha = 0, \\ \alpha = \log(\lambda\,\xi) &\text{ satisfying } \alpha'' + (\alpha')^2 = 0, \end{aligned}$$

where $'$ denotes the derivative with respect to ξ.

Hence, consider the following *elementary system*:

$$\dot{\xi} = \alpha(\xi), \qquad y = \beta(\xi), \tag{9}$$

where $\xi = [\ \xi_1 \ \cdots \ \xi_m\ ]^\top$ is the state vector, $y \in \mathbb{R}$ is the output, $\alpha \in \mathbb{E}^m[\xi]$ and $\beta \in \mathbb{E}[\xi]$. Next theorem states that system (9) can be immersed into polynomial form.

Theorem 2 (Immersion of elementary systems, [27])**.** *Consider system* (9)*, let $\mathcal{D} \subset \mathbb{R}^m$ be the domain in which the elementary functions α and β are defined and smooth, and assume that $\mathcal{D} \neq \emptyset$. Thus, there exists a smooth and injective state immersion $x = \Phi(\xi)$, $\Phi : \mathcal{D} \to \mathbb{R}^n$, globally defined in $\mathcal{D}$, that recasts* (9) *into the polynomial form* (1).

Theorem 2 paves the way toward the use of the tools given in Section II for the observability analysis of elementary systems. As a matter of fact, given system (9), it suffices to first compute the immersion Φ that recasts it into the polynomial form and, secondly, apply the tools given in Section II to study the observability of the latter system.

Example 2. Consider the mechanical system depicted in Figure 1, which is constituted by a mass connected to a frictionless pivot through a rigid and massless link and by a torsion spring, whose rest position is $\theta = 0$.

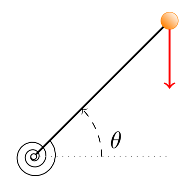

Fig. 1: A pendulum with a torsion spring.

Assuming, for simplicity, unitary values of the physical constants, the dynamics of this system are given by

$$\dot{\xi}_1 = \xi_2, \qquad \dot{\xi}_2 = -\xi_1 - \cos(\xi_1), \tag{10}$$

where ξ_1 denotes the angular displacement of the link and ξ_2 denotes its angular speed. Let the measured output be $y = \xi_2$. Since system (10) is elementary, it can be immersed into a polynomial one. Indeed, the diffeomorphism

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \Phi(\xi) := \begin{bmatrix} \xi_1 \\ \xi_2 \\ \cos(\xi_1) \\ \sin(\xi_1) \end{bmatrix}$$

recasts system (10) into the following polynomial form:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - x_3, \tag{11a}$$
$$\dot{x}_3 = -x_2 x_4, \qquad \dot{x}_4 = x_2 x_3, \qquad y = x_1. \tag{11b}$$

Thus, the tools given in Section II can be used to characterize the observability of this polynomial system. In particular, letting $N = 3$, define the ideal $\mathcal{I}_3$ as in (4). Hence, by computing the reduced Gröbner basis of $\mathcal{I}_3$ with respect to the Lex order, with $x_1 \succ x_2 \succ x_3 \succ x_4 \succ y^{(0)} \succ y^{(1)} \succ y^{(2)} \succ y^{(3)}$, and retaining only the entries that are independent of x, one obtains that $\mathcal{E}_3 = \langle\emptyset\rangle$, *i.e.*, the Zariski closure of $O_3(\mathbb{R}^4)$, where $O_3(\cdot)$ is the observability map of the polynomial system (11), is the whole $\mathbb{R}^4 = \mathbf{V}(\langle\emptyset\rangle)$. Thus let $\mathcal{R}_c = \mathbb{R}[y_{e,3}]/\mathcal{E}_3 = \mathbb{R}[y_{e,3}]$ and let $\mathbb{K} = \mathbb{R}(y_{e,3})$, *i.e.*, the field of all the rational functions with real coefficients in $y_{e,3}$. Coercing $\mathcal{I}_3$ into $\mathbb{K}[x]$, letting $\mathcal{K}_{3,j}$

be defined as in (5), and computing its reduced Gröbner basis $\mathcal{G}_{\mathcal{K}_{3,j}} = \{g_{3,j}\}$, $j = 1, \ldots, 4$, one obtains

$$\begin{aligned} g_{3,1} &= x_1 - \tfrac{y^{(1)}y^{(2)} - (y^{(0)})^3 y^{(1)} - y^{(0)}y^{(3)}}{(y^{(0)})^3}, \\ g_{3,2} &= x_2 - y^{(0)}, \\ g_{3,3} &= x_3 - \tfrac{y^{(0)}y^{(3)} - y^{(1)}y^{(2)}}{(y^{(0)})^3}, \\ g_{3,4} &= x_4 - \tfrac{y^{(0)} + y^{(2)}}{y^{(0)}}. \end{aligned}$$

Therefore, since $\xi_1 = x_1$ and $\xi_2 = x_2$, whenever $y^{(0)} \neq 0$, the following rational expression can be used to express the state ξ of system (10) as a function of $y_{e,3}$:

$$\xi_1 = \tfrac{y^{(1)}y^{(2)} - (y^{(0)})^3 y^{(1)} - y^{(0)}y^{(3)}}{(y^{(0)})^3}, \qquad \xi_2 = y^{(0)}.$$

The main objective of the remainder of this section is to show that a similar construction can be carried out also in the case of systems having discontinuous right-hand side [19].

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a monotonically increasing elementary function such that $\psi(s) \simeq -1$ if $s \ll -1$, and $\psi(s) \simeq 1$, if $s \gg 1$ (*i.e.*, $\psi(s)$ is approximatively -1 for small values of s and approximatively 1 for large values of s), usually referred to as *transition function* [28], [29], [30]. For instance $\psi(s)$ can be chosen either as $\psi(s) = \frac{\exp(s)-1}{\exp(s)+1}$ or as $\psi(s) = \frac{2}{\pi}\arctan(s)$. Hence, letting $\varpi : \mathbb{R}^m \to \mathbb{R}$ be an elementary function having 0 as *regular value* (*i.e.*, if ξ° is such that $\varpi(\xi^\circ) = 0$, then $\frac{\partial \varpi}{\partial \xi}(\xi^\circ) \neq 0$), consider the vector field

$$F(\xi) = \begin{cases} \alpha_1(\xi), & \text{if } \varpi(\xi) \geqslant 0, \\ \alpha_2(\xi) & \text{if } \varpi(\xi) \leqslant 0, \end{cases}$$

where $\alpha_1, \alpha_2 \in \mathbb{E}[\xi]$. The ψ*-regularization* of $F(\xi)$ is the 1-parameter family of vector fields

$$R(\xi, \mu) = (\tfrac{1}{2} + \tfrac{1}{2}\psi(\mu\,\varpi(\xi)))\alpha_1(\xi) + (\tfrac{1}{2} - \tfrac{1}{2}\psi(\mu\,\varpi(\xi)))\alpha_2(\xi),$$

where $\mu \in \mathbb{R}$. Hence, consider the system (which has discontinuous right-hand side)

$$\dot{\xi} \in F(\xi), \qquad y = Z(\xi) \tag{12}$$

and the elementary system (note that the vector field $R(\xi, \mu)$ is elementary due to the fact that the class of these functions is closed with respect to function composition)

$$\dot{\xi} = R(\xi, \mu), \qquad y = Z(\xi). \tag{13}$$

By Tikhonov's theorem [31], under mild assumptions, the trajectories of system (13) approaches the ones of system (12) as μ tends to $+\infty$, i.e., the trajectories of system (13) approach the ones of the differential inclusion (12). Furthermore, since system (13) is elementary, by Theorem 2, there exists an injective immersion Φ that immerses system (13) into

$$\dot{x} = r(x, \mu), \qquad y = z(x, \mu), \tag{14}$$

where $x \in \mathbb{R}^n$ and $r, z \in \mathbb{R}[x, \mu]$. The tools employed in Section II to analyze the observability of system (1) can be adapted to study the stability of system (14) *independently* of μ. In particular, letting $N \in \mathbb{Z}_{\geqslant n-1}$ be fixed and letting $\mathscr{Y}_N := O_N(\mathbb{R}^n, \mathbb{R})$ be the image through $O_N(x, \mu)$ of $\mathbb{R}^n \times \mathbb{R}$, the main objective of the remainder of this section is to determine whether there exists a rational map $K_N : \mathscr{Y}_N \to \mathbb{R}^n$, independent of μ, such that, letting $O_N(x, \mu)$ be the observability map of system (14), one has

$$x = K_N(O_N(x, \mu)), \quad \forall x \in \mathbb{R}^n, \forall \mu \in \mathbb{R}. \tag{15}$$

The main interest in determining a map K_N such that (15) holds relies on the fact that, if such a map exists and system (13) satisfies the assumptions of Tikhonov's theorem, then such a map can be used to design observers for system (12), although it has discontinuous right-hand side.

Letting $r_\mu(x) = r(x, \mu)$ and $z_\mu(x) = z(x, \mu)$, define the ideal $\mathscr{I}_N$ of $\mathscr{R}_a := \mathbb{R}[x, \mu, y_{e,N}]$,

$$\mathscr{I}_N := \langle y^{(0)} - z_\mu(x), \ldots, y^{(N)} - L_{r_\mu}^N z_\mu(x) \rangle. \tag{16}$$

Hence, define the ideal $\mathscr{E}_N := \mathscr{I}_N \cap \mathbb{R}[y_{e,N}]$, and consider the following proposition.

Proposition 2. *The variety $\mathbf{V}(\mathscr{E}_N)$ equals the Zariski closure $\overline{\mathscr{Y}}_N$ of $\mathscr{Y}_N$ and is an irreducible variety.*

By Proposition 2, the ideal $\mathscr{E}_N$ is the set of all the polynomial relations that hold among the time derivatives $y^{(0)}, \ldots, y^{(N)}$ of the output y and that are independent of μ. Hence, in order to determine whether there exists a map $K_N(y_{e,N})$ such that (15) holds, let $\mathscr{R}_c := \mathbb{R}[y_{e,N}]/\mathscr{E}_N$, define the field $\mathbb{K}$ of the rational functions with numerator and denominator being polynomials in the ring $\mathscr{R}_c$, and define the ring $\mathscr{R}_d := \mathbb{K}[x_1, \ldots, x_n]$. Thus, coerce the ideal $\mathscr{I}_N$ into $\mathscr{R}_d$ and consider the elimination ideals

$$\mathscr{K}_{N,j} := \mathscr{I}_N \cap \mathbb{K}[x_j], \tag{17}$$

whose reduced Gröbner basis is $\{\zeta_{N,j}\}$, $\zeta_{N,j} \in \mathbb{K}[x_j]$, by the same reasoning used in Section II, $j = 1, \ldots, n$. The following theorem shows how the polynomials $\zeta_{N,j}$ can be used to characterize the existence of a rational map $K_N(y_{e,N})$ such that (14) holds.

Theorem 3. *There exists a rational map such that* (14) *holds for "almost all"* $(x, \mu) \in \mathbb{R}^n \times \mathbb{R}$ *if and only if*

$$\mathrm{LT}(\zeta_{N,j}) = x_j, \quad j = 1, \ldots, n. \tag{18}$$

Theorem 3 provides a computational tool to verify whether there exists a left-inverse of the observability map of system (14) that is independent of the parameter μ. If such an inverse exists, then it can be used to design observers for system (14) that are independent of μ. In particular, an estimate of the state of system (14) is given by

$$\hat{x}(t) = K_N(\hat{y}_{e,N}(t)), \tag{19}$$

where $\hat{y}_{e,N}$ is an estimate of $y_{e,N}$. Note that also the estimator for $\hat{y}_{e,N}$ can be designed independently of the parameter μ (*e.g.*, by using the high-gain observer given in [13]), thus allowing to estimate the state of system (14) also in the case that it is used to approximate the solutions to system (12) (*i.e.*, when $\mu \to +\infty$).

IV. Application to a physical system

In this section, the techniques outlined in Sections II and III are used to design an observer for a physical system.

Consider the mechanical system depicted in Figure 2, which is constituted by a body moving horizontally and by a spring whose stiffness doubles when compressed.

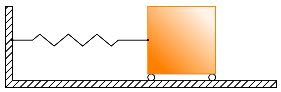

Fig. 2: A mechanical system with a spring.

Assuming unitary values of the physical constants, the dynamics of this plant are given by

$$\dot{\xi} \in F(\xi) := \begin{cases} [\ \xi_2 \quad -\xi_1\ ]^\top, & \text{if } x_1 \geqslant 0, \\ [\ \xi_2 \quad -2\xi_1\ ]^\top, & \text{if } x_1 \leqslant 0, \end{cases} \tag{20}$$

where ξ_1 denotes the position of the body with respect to the rest point of the spring and ξ_2 denotes its velocity. Let the available measure be $y = \xi_2$. Thus, letting $\psi(s) = \frac{\exp(s)-1}{\exp(s)+1}$, consider the ψ-regularization $R(\xi, \mu)$ of $F(\xi)$,

$$R(\xi, \mu) = \begin{bmatrix} \xi_2 \\ \xi_1 \left(-\frac{1}{\exp(\mu\,\xi_1)+1} - 1 \right) \end{bmatrix}.$$

Since $R(\xi, \mu) \in \mathbb{E}[\xi, \mu]$, there exists an injective immersion $\Phi(\xi)$ that recast the system in polynomial form. Indeed, consider the immersion

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \Phi(\xi) := \begin{bmatrix} \xi_1 \\ \xi_2 \\ \tanh\left(\frac{\mu\,\xi_1}{2}\right) \end{bmatrix}.$$

Such an immersion recasts the elementary system $\dot{x} = R(x, \mu)$ into the parametric polynomial form

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \tfrac{1}{2}x_1 x_3 - \tfrac{3}{2}x_1, \tag{21a}$$

$$\dot{x}_3 = -\tfrac{1}{2}\mu x_2 x_3^2 + \tfrac{1}{2}\mu x_2, \qquad y = x_2. \tag{21b}$$

Thus, the tools given in Section III can be used to determine whether there exists a rational map such that (15) holds. In practice, let $N = 4$ and define the ideal $\mathscr{I}_4$ as in (16). Hence, by computing the reduced Gröbner basis of $\mathscr{I}_4$ with respect to the Lex order, with $x_1 \succ x_2 \succ x_3 \succ y^{(0)} \succ y^{(1)} \succ y^{(2)} \succ y^{(3)} \succ y^{(4)}$, and retaining only the entries that are independent of x, one obtains that $\mathscr{E}_4 = \langle \eta_4 \rangle$, where η_4 is a homogeneous polynomial of degree 14 whose leading term is $256(y^{(0)})^{12}(y^{(4)})^2$ (the explicit expression of this polynomial is omitted for brevity). This implies that the time derivatives up to order 4 of system (21) are algebraically dependent and η_4 is an embedding of system (21).

Let $\mathscr{R}_b = \mathbb{R}[y_{e,4}]$, let $\mathscr{R}_c = \mathscr{R}_b/\mathscr{E}_4$, let $\mathbb{K}$ be the field of all the rational functions whose numerator and denominator are in $\mathscr{R}_c$, and let $\mathscr{R}_d = \mathscr{K}[x]$. Thus, by coercing $\mathscr{I}_4$ into $\mathscr{R}_d$ and computing the reduced Gröbner basis $\mathscr{G}_{\mathscr{K}_{4,j}} = \{g_{4,j}\}$ of $\mathscr{K}_{4,j} = \mathscr{I}_4 \cap \mathbb{K}[x_j]$, $j = 1, \ldots, 3$, one obtains that the condition given in (18) holds. Hence, by Theorem 3, there exists a rational map $K_4(y_{e,4})$ such that (15) holds for "almost all" $(x, \mu) \in \mathbb{R}^3 \times \mathbb{R}$. The polynomials $g_{4,j}$ can be directly used to obtain such a map. In particular, one obtains that x_1 can be expressed as a rational function of $y_{e,4}$ whose numerator is a homogeneous polynomial of degree 9 and whose denominator is a homogeneous polynomial of degree 8, whereas x_3 can be expressed as a rational function of $y_{e,4}$ whose numerator and denominator are homogeneous polynomials of degree 8 (the explicit expression of such polynomials is omitted for brevity).

The map $K_4(y_{e,4})$ can be used to design an observer for system (21) that is independent of the parameter μ, and that therefore can be used also to estimate the state of system (20). In particular, a high-gain "practical" observer for the time derivatives of y that does not require the knowledge of the parameter μ is given by [13],

$$\dot{\hat{y}}_{e,4} = \begin{bmatrix} -k_1\varepsilon^{-1} & 1 & 0 & 0 & 0 \\ -k_2\varepsilon^{-2} & 0 & 1 & 0 & 0 \\ -k_3\varepsilon^{-3} & 0 & 0 & 1 & 0 \\ -k_4\varepsilon^{-4} & 0 & 0 & 0 & 1 \\ -k_5\varepsilon^{-5} & 0 & 0 & 0 & 0 \end{bmatrix} \hat{y}_{e,4} + \begin{bmatrix} k_1\varepsilon^{-1} \\ k_2\varepsilon^{-2} \\ k_3\varepsilon^{-3} \\ k_4\varepsilon^{-4} \\ k_5\varepsilon^{-5} \end{bmatrix} y, \tag{22}$$

where $k_1, \ldots, k_5$ are such that the roots of the polynomial $\lambda^5 + k_1\lambda^4 + \cdots + k_5$ have negative real part, $\varepsilon > 0$ is a sufficiently small parameter, and $\hat{y}_{e,4}$ is an estimate of $y_{e,4}$. By coupling system (22) with the expression given in (19), it is possible to obtain an estimate $\hat{x}$ of the state x of system (21). Finally, an estimate $\hat{\xi}$ of the state ξ of system (9) is $\hat{\xi} = \Phi^{-1}(\hat{x})$.

A numerical simulation has been carried out to test the proposed design strategy to estimate the state of system (20). In particular, letting y be the output of system (20), the high-gain observer (22) has been used to estimate the time derivatives of the signal y up to order 4, while the map $K_N(y_{e,4})$, coupled with the inverse of the immersion Φ,

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \Phi^{-1}(x) := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

has been used to obtain the estimate $\hat{\xi} = \Phi^{-1}(K_N(\hat{y}_{e,4}))$ of the state of system (20). Figure 3 depicts the results of such a numerical simulation in which the following parameters have been taken: $\xi(0) = [\ 1 \quad 3\ ]^\top$, $\hat{y}_{e,4}(0) = 0$, $k_1 = 5$, $k_2 = 10$, $k_3 = 10$, $k_4 = 5$, $k_5 = 1$, and $\varepsilon = 10^{-3}$.

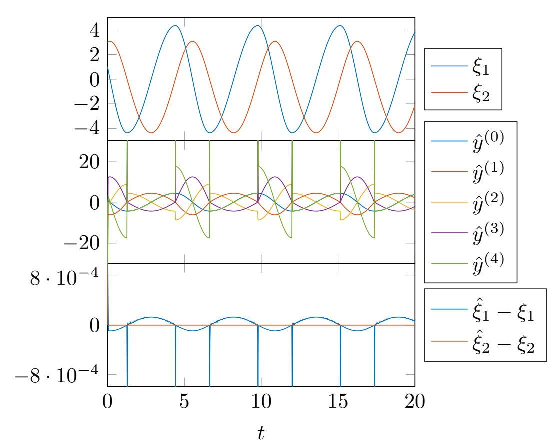

Fig. 3: Results of the numerical simulation.

Apart from small transient errors that occur for each time t at which $\xi_2(t) = 0$ (due to the fact that the signal $y_{e,4}(t)$

is discontinuous at such times, whereas, by construction, the signal $\hat{y}_{e,4}(t)$ is continuous), the proposed observer is able to "practically" reconstruct the state of system (20). Indeed, by [13], the high-gain observer (22) is such that the estimation error $y_{e,4}-\hat{y}_{e,4}$ can be made arbitrarily small in an arbitrarily small amount of time, by letting the parameter $\varepsilon > 0$ be sufficiently small. Thus, since the rational function $K_4(y_{e,4})$ is absolutely continuous in its domain and $x = K_4(y_{e,4})$ for "almost all" $y_{e,4} \in \mathscr{Y}_4$, the error $\xi - \hat{\xi}$ can be made arbitrarily small in an arbitrarily small amount of time by letting $\varepsilon > 0$ be sufficiently small.

CONCLUDING REMARKS

In this paper, it has been shown that algebraic geometry tools can be used to characterize the observability and to design observers for a class of continuous-time dynamical systems having discontinuous right-hand side. In particular, it has been shown that such tools can be efficiently employed to determine whether there exists a rational inverse of the observability map of a polynomial system. Hence, by showing that many systems arising from practical application can be recast into polynomial form through an injective immersion and that several systems having discontinuous right-hand side can be approximated by a 1-parameter family of elementary systems, the tools developed for polynomial plants are adapted to determine whether there exists a rational inverse of the observability map of the 1-parameter family that is independent of the parameter governing the approximation.

It is worth noticing that the proposed methods allow one also to design observers for polynomial, elementary and discontinuous systems. As a matter of fact, a state observer for these systems can be designed by coupling any tool that is able to estimate the time derivatives of the output of a system (such as high gain observers [13], [15], [32], sliding mode tools [18], and super twisting algorithms [26]) with a left inverse of the observability map, which can be computed directly by using the methods given in this paper.

Note that, in order to apply the proposed method, one has to check whether the condition given in (18) is satisfied for $N \in \{n-1, n, n+1, \ldots, \bar{N}\}$, where $\bar{N}$ is a sufficiently large integer. Therefore, the drawback of the proposed technique is that the (off-line) computational complexity of the proposed method may be large if the integer $\bar{N}$ is large.

Furthermore, it is worth mentioning that, although computing the Gröbner bases is an EXPSPACE-complete problem [33], the proposed observation scheme requires to compute such bases just once and off-line. Hence, it can be readily used to design computationally efficient observers for nonlinear plants having discontinuities.

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